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# Affine Toda–Sutherland systems

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**Abstract**

A cross between two well-known integrable multi-particle dynamics, an affine Toda molecule and a Sutherland system, is introduced for any affine root system. Though it is not completely integrable but partially integrable, or quasi-exactly solvable, it inherits many remarkable properties from the parents. The equilibrium position is algebraic, i.e. proportional to the Weyl vector. The frequencies of small oscillations near equilibrium are proportional to the affine Toda masses, which are essential ingredients of the exact factorizable $S$-matrices of affine Toda field theories. Some lower lying frequencies are *integer* times a coupling constant for which the corresponding exact quantum eigenvalues and eigenfunctions are obtained. An affine Toda–Calogero system, with a corresponding rational potential, is also discussed.

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## 1. Introduction

Calogero–Moser systems and (affine) Toda molecules[4] are the best known examples of integrable/solvable many-particle dynamics on a line which are based on root systems. The original Toda model [1] and the Calogero [2] and the Sutherland [3] models are based on the $A_r$ root system which corresponds to the Lie algebra $\mathfrak{su}(r+1)$. Later integrable Toda [4, 5] and Calogero–Moser (CM) [5–9] systems are formulated for any root system. The potentials of Toda systems are exponential functions of the coordinates, whereas those of Calogero–Moser systems are rational $1/q^2$, trigonometric $1/\sin^2 q$, hyperbolic $1/\sinh^2 q$ and elliptic $\wp(q)$ functions, in which $\wp$ is the Weierstrass function and $q$ denotes the coordinates generically. In the CM systems, the elliptic potentials are the most general ones and the rest (trigonometric, hyperbolic and rational) are obtained by various degenerations. In fact, a Toda molecule is obtained from an elliptic CM system by a special limiting procedure [10–12]. While the

[4] In this paper we use the terminology 'molecule' to emphasize the finite degrees of freedom instead of the more familiar 'lattice' which might be misinterpreted as an infinitely or macroscopically large system.

  

potential of a CM system depends on *all* (positive) roots, that of an (affine) Toda system contains (affine) *simple* roots only. For the $A$-type root systems the above feature is usually referred to as that the CM potential gives a *pairwise* interactions and the Toda potential is of the *nearest neighbour* interaction type and the affine simple root corresponds to a *periodic* boundary condition.

In this paper, we will present two new types of multi-particle dynamics related to any root system. Roughly speaking, each could be considered as a cross between an (affine) Toda molecule and a CM system. The first, to be tentatively called an (affine) Toda–Sutherland system, has trigonometric potentials $1/\sin^2 q$ and depends on the (affine) *simple* roots only. The second, to be tentatively referred to as an (affine) Toda–Calogero system, has rational potentials $1/q^2$ plus a harmonic confining potential $q^2$ and depends on the (affine) *simple* roots only. The former has much richer structure than the latter and in this paper we mainly discuss the affine Toda–Sutherland systems. We do not think that they are integrable, either at the classical or the quantum level. But they have many remarkable features as shown in some detail for the systems based on the $A$-type root systems [13–15]. Their potentials have the *nearest* and *next-to-nearest* neighbour interactions, in contrast to the nearest neighbour interactions of the (affine) Toda molecule. These dynamical systems exhibit a behaviour intermediate between regular and chaotic. Like the CM systems, these multi-particle dynamics are closely related to random matrix theory [13].

At the classical level, the frequencies of small oscillations at equilibrium [16] of an affine Toda–Sutherland system have exactly the same pattern as those of the affine Toda molecule based on the same root system. Let us point out that the pattern of the frequencies of small oscillations at equilibrium of an affine Toda molecule, or the so-called *affine Toda masses* appearing in the affine Toda field theory in $1+1$ dimensions [17], is the essential ingredient for its *exact factorizable* $S$-matrices. At the quantum level, most (but not all) of the multi-particle systems discussed in this paper are *quasi-exactly solvable* (QES) [18, 19]. That is, on top of the ground state eigenfunctions, a certain small number of eigenvalues and eigenfunctions is obtained exactly. The mechanism for QES seems very different from that of known ones [18–20].

This paper is organized as follows. In section 2, the salient features of affine Toda molecules are reviewed with a brief summary of roots and weights as essential ingredients. Section 3 is the main body of the paper. In section 3.1 we obtain the frequencies of small oscillations for Toda–Sutherland systems based on affine root systems. For these multi-particle systems we present some exact eigenvalues and eigenfunctions in section 3.2. They correspond to the low lying *integer* (times a coupling constant) frequencies of the small oscillations at equilibrium [16, 21]. In section 4 the affine Toda–Calogero systems are briefly discussed. The final section is reserved for summary and comments. In this paper, we adopt the convention that $\hbar = 1$ and do not show the dependence on Planck's constant.

## 2. Affine Toda molecule

The dynamical variables of a classical (quantum) multi-particle system to be discussed in this paper, an (affine) Toda molecule, a CM system, an (affine) Toda–Sutherland system and an (affine) Toda–Calogero system, are the coordinates $(\{q_j | j = 1, \ldots, r\})$ and their canonically conjugate momenta $(\{p_j | j = 1, \ldots, r\})$, with the Poisson bracket (Heisenberg commutation) relations:

$$\{q_j, p_k\} = \delta_{jk} \qquad \{q_j, q_k\} = \{p_j, p_k\} = 0 \qquad [q_j, p_k] = \mathrm{i}\delta_{jk} \qquad [q_j, q_k] = [p_j, p_k] = 0.$$

These will be denoted by vectors in $\mathbf{R}^r$

$$q = (q_1, \ldots, q_r) \qquad p = (p_1, \ldots, p_r)$$

in which $r$ is the number of particles and it is also the rank of the underlying root system $\Delta$.

### *2.1. Roots and weights*

Let $\Pi$ be the set of simple roots of $\Delta$:

$$\Pi = (\alpha_1, \alpha_2, \ldots, \alpha_r). \tag{2.1}$$

Any positive roots in $\Delta$ can be expressed as a linear combination of the simple roots with non-negative integer coefficients

$$\alpha = \sum_{j=1}^{r} m_j \alpha_j \qquad m_j \in \mathbb{Z}_+ \quad \forall \alpha \in \Delta_+. \tag{2.2}$$

In the case of *simply laced* root systems ($A$, $D$, $E$) all the roots have the same length. We adopt the convention $\alpha^2 = \alpha \cdot \alpha = 2$. In the case of *non-simply laced* root systems ($B$, $C$, $F_4$, $G_2$), there are long roots and short roots. We adopt the convention $\alpha_L^2 = 2$ except for the $C$-series of the root system in which we adopt $\alpha_S^2 = 2$. Since $\Delta$ is a finite set, there exists an element $\alpha_h$ for which $\sum_{j=1}^{r} m_j$ is the maximum in $\Delta_+$. We call it the *highest root* and write it

$$\alpha_h = \sum_{j=1}^{r} n_j \alpha_j \qquad n_j \in \mathbb{Z}_+. \tag{2.3}$$

For the non-simply laced root systems, the highest roots are always long. We also introduce the *highest short root* and denote it in the same way as (2.3) to avoid duplicating many formulae.

The positive integers $\{n_j\}$ are called *Dynkin–Kac labels*. We define the affine simple root $\alpha_0$ as the *lowest* (*short*) *root*, that is the negative of the highest (short) root:

$$\alpha_0 = -\alpha_h = -\left( \sum_{j=1}^{r} n_j \alpha_j \right). \tag{2.4}$$

The above relationship can be rewritten in a symmetrical way:

$$\sum_{j=0}^{r} n_j \alpha_j = 0 \qquad n_0 \equiv 1. \tag{2.5}$$

We call $\Pi_0$ the set of *affine simple roots*,

$$\Pi_0 = \alpha_0 \cup \Pi = (\alpha_0, \alpha_1, \ldots, \alpha_r) \tag{2.6}$$

which specifies the *affine Lie algebra*, to be denoted as $A_r^{(1)}$, $E_6^{(2)}$, $D_4^{(3)}$, etc. It has the necessary and sufficient information for defining the affine Toda molecule (and its field theory version, the affine Toda field theory [17], too).

The fundamental weights $\{\lambda_j\}$ are the dual to the simple roots:

$$\alpha_j^\vee \cdot \lambda_k = \delta_{jk} \qquad \alpha_j^\vee \equiv \frac{2\alpha_j}{\alpha_j^2} \qquad j = 1, \ldots, r \tag{2.7}$$

$$\alpha_j \cdot \lambda_j^\vee = \delta_{jk} \qquad \lambda_j^\vee \equiv \frac{2}{\alpha_j^2} \lambda_j \qquad j = 1, \ldots, r. \tag{2.8}$$

Equation (2.7) defines $\{\lambda_j\}$ and (2.8) defines $\left\{\lambda_j^\vee\right\}$ in turn. Next we define $\varrho$,

$$\varrho \equiv \sum_{j=1}^{r} \lambda_j^\vee \tag{2.9}$$

which is essentially the *Weyl vector* having the following properties,

$$\alpha_j \cdot \varrho = 1 \qquad j = 1, \ldots, r \tag{2.10}$$

$$\alpha_0 \cdot \varrho = -\left(\sum_{j=1}^{r} n_j \alpha_j\right) \cdot \varrho = -\left(\sum_{j=1}^{r} n_j\right) = -(h-1) \tag{2.11}$$

in which $h$ is the (dual) *Coxeter number*:

$$h \equiv \sum_{j=0}^{r} n_j = 1 + \sum_{j=1}^{r} n_j. \tag{2.12}$$

*2.2. Hamiltonian, equilibrium position and frequencies of small oscillations*

The Hamiltonian of the affine Toda molecule based on the set of affine simple roots $\Pi_0$ is

$$H = \tfrac{1}{2} p^2 + V(q) \tag{2.13}$$

$$V(q) = \frac{1}{\beta^2} \sum_{j=0}^{r} n_j \, \mathrm{e}^{\beta \alpha_j \cdot q} \tag{2.14}$$

in which $\beta \in \mathbf{R}$ is the coupling constant. Note that all the particle masses are the same and normalized to unity. The potential $V(q)$ has a minimum (*equilibrium point*) at $q = 0$ as

$$V(q) = \frac{h}{\beta^2} + \frac{1}{\beta}\left(\sum_{j=0}^{r} n_j \alpha_j\right) \cdot q + \frac{1}{2} \sum_{j=0}^{r} \sum_{k,l}^{r} n_j (\alpha_j)_k (\alpha_j)_l q_k q_l + o(q^3). \tag{2.15}$$

Here, $(\alpha_j)_k$ is the $k$th component of the (affine) simple root $\alpha_j$. The linear term vanishes due to (2.5) and the constant term is proportional to the Coxeter number $h$ given by (2.12).

The symmetric matrix $M$

$$M_{kl} = \sum_{j=0}^{r} n_j (\alpha_j)_k (\alpha_j)_l \qquad \text{or} \qquad M = \sum_{j=0}^{r} n_j \alpha_j \otimes \alpha_j \tag{2.16}$$

is called the *affine Toda mass matrix*. Its eigenvalues

$$\mathrm{Spec}(M) = \left\{m_1^2, m_2^2, \ldots, m_r^2\right\} \qquad m_j > 0 \tag{2.17}$$

are called affine Toda masses (squared). The set $\{m_1, m_2, \ldots, m_r\}$ gives $r$ (angular) frequencies of small oscillations at the equilibrium $q = 0$. The above Hamiltonian (2.13), (2.14) is *completely integrable* and the *classical* Lax pair is known for all the affine simple root systems. This is a *periodic* Toda lattice if $\Pi_0$ is for $A_r^{(1)}$.

## 3. Affine Toda–Sutherland systems

The multi-particle dynamics with nearest and next-to-nearest trigonometric interactions introduced in [13, 14] can be called the *affine Toda–Sutherland model* based on $A_r^{(1)}$. They can be generalized to any root system as follows.

Given an affine root system $\Pi_0$, let us introduce a *prepotential* $W$

$$W(q) = \beta \sum_{j=0}^{r} n_j \log|\sin(\alpha_j \cdot q)| \tag{3.1}$$

in which $\beta \in \mathbf{R}_+$ is a positive coupling constant and $\{n_j\}$ are the Dynkin–Kac labels for $\Pi_0$. This leads to the Hamiltonians with the classical and quantum potentials $V_C$ and $V_Q$ as [9]

$$H_C = \frac{1}{2}p^2 + V_C(q) \qquad V_C(q) = \frac{1}{2}\sum_{j=1}^{r}\left(\frac{\partial W}{\partial q_j}\right)^2 \tag{3.2}$$

$$H_Q = \frac{1}{2}p^2 + V_Q(q) \qquad V_Q(q) = \frac{1}{2}\sum_{j=1}^{r}\left[\left(\frac{\partial W}{\partial q_j}\right)^2 + \frac{\partial^2 W}{\partial q_j^2}\right]. \tag{3.3}$$

Again note that all the particle masses are the same and normalized to unity. Explicitly $V_Q$ reads

$$V_Q = \frac{1}{2}\sum_{j=0}^{r}\frac{\beta n_j(\beta n_j - 1)\alpha_j^2}{\sin^2(\alpha_j \cdot q)} + \beta^2\sum_{j<k} n_j n_k \alpha_j \cdot \alpha_k \cot(\alpha_j \cdot q)\cot(\alpha_k \cdot q) - E_0 \tag{3.4}$$

$$E_0 = \frac{\beta^2}{2}\sum_{j=0}^{r} n_j^2 \alpha_j^2 \tag{3.5}$$

in which the constant part $E_0$ can be considered as the *ground state energy*. The extended Dynkin diagram of $\Pi_0$ encodes all the necessary information $\{\alpha_j^2\}$, $\{\alpha_j \cdot \alpha_k\}$ and $\{n_j\}$ to determine $V_Q$. See [9, 16, 21] for the formulation of Hamiltonian dynamics in terms of a prepotential and the frequencies of small oscillations at equilibrium. The corresponding ground state wavefunction is

$$H_Q\psi_0 = 0 \qquad \psi_0(q) = \mathrm{e}^{W(q)} = \prod_{j=0}^{r}|\sin(\alpha_j \cdot q)|^{\beta n_j}. \tag{3.6}$$

In contrast to the Calogero–Moser systems [7, 8], the prepotential (3.1), potential (3.3) and thus the Hamiltonian are not Weyl invariant. For simplicity, we consider the configuration space in the *principal Weyl alcove*:

$$\mathrm{PW}_T = \{q \in \mathbf{R}^r | \alpha \cdot q > 0, \quad \alpha \in \Pi, \quad \alpha_h \cdot q < \pi\} \tag{3.7}$$

where $(\alpha_h)$ is the highest root. (Due to the non-invariance under the Weyl group, theories with different configuration spaces are physically different. For example, they have different (non-equivalent) equilibrium positions.)

For the simplest affine Lie algebra of $A_r^{(1)}$ the quantum Hamiltonian reads[5]

$$H_Q = \frac{1}{2}p^2 + \beta(\beta - 1)\sum_{j=1}^{r+1}\frac{1}{\sin^2(q_j - q_{j+1})} - \beta^2\sum_{j=1}^{r+1}\cot(q_{j-1} - q_j)\cot(q_j - q_{j+1}) - \beta^2(r+1). \tag{3.8}$$

This has the *nearest* and *next-to-nearest* neighbour interactions [13, 14]. The $B$, $BC$ and $D$ models in [13–15] are different from those in this paper.

[5] For $A_r$ models, it is customary to introduce one more degree of freedom, $q_{r+1}$ and $p_{r+1}$ and embed all of the roots in $\mathbf{R}^{r+1}$. Here we also adopt the 'periodic' convention, $q_{r+1} \equiv q_0$, $q_{r+2} \equiv q_1$, etc.

*3.1. Classical equilibrium*

The equilibrium point $(\bar{q})$ of the classical Hamiltonian of the affine Toda–Sutherland system

$$H = \frac{1}{2}p^2 + V_C(q) \qquad V_C(q) = \frac{1}{2}\sum_{j=1}^{r}\left(\frac{\partial W}{\partial q_j}\right)^2 \tag{3.9}$$

$$\frac{\partial W}{\partial q_j} = \beta \sum_{k=0}^{r} n_k (\alpha_k)_j \cot[\alpha_k \cdot q] \qquad j = 1, \ldots, r \tag{3.10}$$

has a very intuitive characterization. It is proportional to the Weyl vector $\varrho$ (2.9), $\bar{q} \propto \varrho$, the fundamental quantity of the Lie algebra. This is much simpler than the cases in the Calogero as well as Sutherland systems in which $\bar{q}$ correspond to the zeros of certain polynomials, i.e. the Hermite, Laguerre, Chebyshev and Jacobi polynomials for classical root systems [22, 16]. Since [16, 21]

$$\frac{\partial W(\bar{q})}{\partial q_j} = 0 \quad j = 1, \ldots, r \quad \Rightarrow \quad \frac{\partial V_C(\bar{q})}{\partial q_l} = \sum_{j=1}^{r} \frac{\partial^2 W(\bar{q})}{\partial q_j \partial q_l} \frac{\partial W(\bar{q})}{\partial q_j} = 0 \tag{3.11}$$

the equilibrium is achieved at the point $\bar{q}$ where all $\partial W/\partial q_j$ vanish, i.e. at the maximum of the ground state wavefunction. It is easy to see that

$$\bar{q} = c\varrho \qquad c : \text{const} \tag{3.12}$$

gives a solution. Using (2.9)–(2.11),

$$\alpha_k \cdot \bar{q} = \begin{cases} c & k = 1, \ldots, r \\ -(h-1)c & k = 0 \end{cases} \tag{3.13}$$

we obtain

$$\frac{\partial W(\bar{q})}{\partial q_j} = \beta \left( \cot(c) \sum_{k=1}^{r} n_k (\alpha_k)_j - \cot[(h-1)c](\alpha_0)_j \right). \tag{3.14}$$

For

$$c = \frac{\pi}{h} \tag{3.15}$$

$c\varrho$ is in the principal Weyl alcove (3.7) and

$$\cot[(h-1)c] = \cot(\pi - c) = -\cot(c). \tag{3.16}$$

Thus we find $\bar{q} = \pi\varrho/h$ is the equilibrium

$$\frac{\partial W(\bar{q})}{\partial q_j} = \beta \cot\left[\frac{\pi}{h}\right] \left( \sum_{k=0}^{r} n_k \alpha_k \right)_j = 0. \tag{3.17}$$

The equilibrium points are *equally spaced* for all the classical root systems. The situation is different for the exceptional root systems. The equilibrium point $\bar{q} = \pi\varrho/h$ is unique in the principle Weyl alcove (3.7).

The squared frequencies of small oscillations at equilibrium $\bar{q}$ are given by the eigenvalues of the matrix

$$\left.\frac{\partial^2 V_C(q)}{\partial q_j \partial q_k}\right|_{\bar{q}} = \sum_{j=1}^{r} \left.\frac{\partial^2 W(q)}{\partial q_j \partial q_l}\right|_{\bar{q}} \left.\frac{\partial^2 W(q)}{\partial q_l \partial q_k}\right|_{\bar{q}} = (\widetilde{W}^2)_{jk}. \tag{3.18}$$

**Table 1.** The Coxeter number $h$ and the affine Toda masses $m_j^2$ for classical untwisted affine Lie algebras.

| $\Pi_0$ | $h$ | Affine Toda masses |
|---|---|---|
| $A_r^{(1)}$ | $r+1$ | $m_j^2 = 4\sin^2\left(\frac{j\pi}{h}\right), \quad j = 1, \ldots, r$ |
| $B_r^{(1)}$ | $2r$ | $m_j^2 = 8\sin^2\left(\frac{j\pi}{h}\right), \quad j = 1, \ldots, r-1, \quad m_r^2 = 2$ |
| $C_r^{(1)}$ | $2r$ | $m_j^2 = 8\sin^2\left(\frac{j\pi}{h}\right), \quad j = 1, \ldots, r$ |
| $D_r^{(1)}$ | $2(r-1)$ | $m_j^2 = 8\sin^2\left(\frac{j\pi}{h}\right), \quad j = 1, \ldots, r-2, \quad m_{r-1}^2 = m_r^2 = 2$ |

Thus the frequencies of small oscillations at equilibrium $\bar{q}$ are given by the eigenvalues of a symmetric matrix $\widetilde{W}$ defined by

$$\widetilde{W}_{jk} = -\left.\frac{\partial^2 W(q)}{\partial q_j \partial q_k}\right|_{\bar{q}} = \frac{\beta}{\sin^2\frac{\pi}{h}} \sum_{l=0}^{r} n_l (\alpha_l)_j (\alpha_l)_k = \frac{\beta}{\sin^2\frac{\pi}{h}} M_{jk} \tag{3.19}$$

in which matrix $M$ is the mass square matrix of the affine Toda molecule associated with the affine root system $\Pi_0$ defined in (2.16).

The frequencies (not frequencies squared) of small oscillations at equilibrium of affine Toda–Sutherland model are given up to the coupling constant $\beta$ by

$$\frac{1}{\sin^2\frac{\pi}{h}} \left\{ m_1^2, m_2^2, \ldots, m_r^2 \right\} \tag{3.20}$$

in which $m_j^2$ are the affine Toda masses. In [17] it is shown that the vector $\mathbf{m} = (m_1, \ldots, m_r)$, if ordered properly, is the *Perron–Frobenius* eigenvector of the incidence matrix (the Cartan matrix) of the corresponding root system. Therefore there exists a one-to-one correspondence between the mass $m_j$ and a vertex (or the fundamental weight) of the Dynkin diagram. This fact will be important in the next subsection for the explicit construction of exact eigenvalues and eigenfunctions. In table 1 we list the affine Toda masses and the Coxeter number $h$ for the classical *untwisted* affine Lie algebras, $A_r^{(1)}$, $B_r^{(1)}$, $C_r^{(1)}$, $D_r^{(1)}$, see [17].

Those for the exceptional affine Lie algebras $E_r^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$ we refer to [17]. (The affine Toda masses for $E_8$ reported there need a factor 2.) The *twisted affine Lie algebras*, for example $D_{r+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$, etc, which are characterized by the highest short roots, can also be obtained from untwisted affine Lie algebras by *folding* [17]. The affine Toda masses for the twisted affine Lie algebra are closely related to those of the original untwisted affine Lie algebra.

### *3.2. Quantum eigenfunctions*

Here we demonstrate that some of the quantum affine Toda–Sutherland (3.3) systems have a number of exact eigenvalues and eigenfunctions and thus they are partially integrable or *quasi-exactly solvable* [18, 19]. These are usually a small number of lowest lying excited states. The occurrence of such exact states is strongly correlated with the appearance of the *integer eigenvalues* in the spectrum of the small oscillations near the *classical equilibrium*, as shown in the recent general theorems by Loris–Sasaki [21]. Let us express the eigenfunctions in product forms

$$\psi_n(q) = \phi_n(q)\psi_0(q) \qquad n = 0, 1, \ldots \quad \phi_0 \equiv 1 \tag{3.21}$$

in which $\phi_n$ obeys a simplified equation with the similarity transformed Hamiltonian $\tilde{H}$ [9]:

$$\tilde{H}\phi_n = E_n \phi_n \tag{3.22}$$

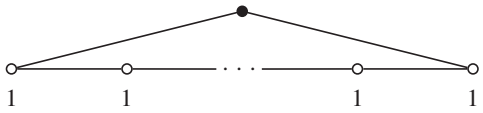


**Figure 1.** $A_r^{(1)}$ Dynkin diagram with the numbers $n_j$ attached. The black spot is the affine simple root.

$$\tilde{H} = \mathrm{e}^{-W} H_Q \, \mathrm{e}^{W} = -\frac{1}{2}\triangle - \sum_{j=1}^{r} \frac{\partial W}{\partial q_j}\frac{\partial}{\partial q_j} \qquad \triangle \equiv \sum_{j=1}^{r} \frac{\partial^2}{\partial q_j^2}. \tag{3.23}$$

*3.2.1.* $A_r^{(1)}$. In this case the spectrum of the small oscillations, up to the coupling constant $\beta$ is easily read from table 1:

$$4\{1, \ldots, \sin^2(j\pi/(r+1))/\sin^2(\pi/(r+1)), \ldots, 1\}. \tag{3.24}$$

Reflecting the left–right mirror symmetry $j \leftrightarrow r+1-j$ of the Dynkin diagram figure 1, the spectrum is doubly degenerate except for the possible singlet at the middle point $j = (r+1)/2$ for odd $r$.

The doubly degenerate integer eigenvalues 4 correspond to the two end points of the $A_r^{(1)}$ Dynkin diagram, figure 1. They correspond to the fundamental *vector* and *conjugate vector* representations and to the eigenfunctions:

$$\mathbf{v} = \sum_{j=1}^{r+1} \mathrm{e}^{2\mathrm{i}q_j} \qquad \bar{\mathbf{v}} = \sum_{j=1}^{r+1} \mathrm{e}^{-2\mathrm{i}q_j}. \tag{3.25}$$

It is easy to verify

$$\tilde{H}\mathbf{v} = (4\beta+2)\mathbf{v} \qquad \tilde{H}\bar{\mathbf{v}} = (4\beta+2)\bar{\mathbf{v}} \qquad -\tfrac{1}{2}\triangle\mathbf{v} = 2\mathbf{v} \qquad -\tfrac{1}{2}\triangle\bar{\mathbf{v}} = 2\bar{\mathbf{v}}. \tag{3.26}$$

The affine simple root corresponds to the adjoint representation. Let us define

$$\phi_a = \phi_{v\bar{v}} + 2\beta/(1+2\beta) \qquad \phi_{v\bar{v}} = \sum_{j\neq k} \mathrm{e}^{2\mathrm{i}(q_j - q_k)}. \tag{3.27}$$

It is easy to show

$$\tilde{H}\phi_a = (8\beta+4)\phi_a \tag{3.28}$$

in which $8\beta$ is simply a sum of $4\beta$ for $\mathbf{v}$ and another $4\beta$ for $\bar{\mathbf{v}}$ in (3.25).

For $A_2^{(1)}$, the system is identical with the $A_2$ Sutherland model. For the special case of $A_3^{(1)}$, the above spectrum (3.24) is $\{4, 8, 4\}$. We find another complex eigenfunction with the classical eigenvalue $8\beta$,

$$\phi_t = \sum_{j=1}^{4} \mathrm{e}^{4\mathrm{i}q_j} - \mathrm{e}^{2\mathrm{i}(q_1+q_2+q_3+q_4)} \sum_{j=1}^{4} \mathrm{e}^{-4\mathrm{i}q_j} \qquad \tilde{H}\phi_t = (8\beta+8)\phi_t. \tag{3.29}$$

*3.2.2.* $D_r^{(1)}$. The spectrum of small oscillations, up to the coupling constant $\beta$, is easily read from table 1,

$$8\{1, \ldots, \sin^2(j\pi/2(r-1))/\sin^2(\pi/2(r-1)), \ldots\} \quad \text{and} \quad 2/\sin^2(\pi/2(r-1))[2] \tag{3.30}$$

in which the two degenerate frequencies at the end correspond to the spinor and anti-spinor weights at the right end of the $D_r^{(1)}$ Dynkin diagram in figure 2.

For $r \geqslant 5$ these eigenvalues are greater than 8, which belongs to the vector weights at the left end of the Dynkin diagram, figure 2. The set of vector weights is $\mathbf{V} = \{\pm\mathbf{e}_j | j = 1, \ldots, r\}$.

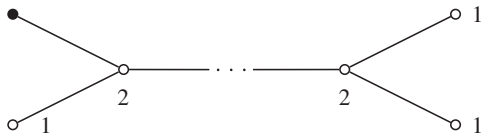


**Figure 2.** $D_r^{(1)}$ Dynkin diagram with the numbers $n_j$ attached. The black spot is the affine simple root.

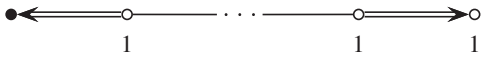


**Figure 3.** $D_{r+1}^{(2)}$ Dynkin diagram with the numbers $n_j$ attached. The black spot is the affine simple root.

Let us introduce the corresponding wavefunctions

$$\phi_{\mathbf{V}} = \sum_{\mu\in\mathbf{V}} \mathrm{e}^{2\mathrm{i}\mu\cdot q} = 2\sum_{j=1}^{r} \cos 2q_j. \tag{3.31}$$

However, it is not an eigenfunction

$$\tilde{H}\phi_{\mathbf{V}} = (8\beta+2)\phi_{\mathbf{V}} - 8\beta(\cos 2q_1 + \cos 2q_r) \qquad -\tfrac{1}{2}\triangle\phi_{\mathbf{V}} = 2\phi_{\mathbf{V}}. \tag{3.32}$$

This would give an eigenfunction in a theory if $q_1$ and $q_r$ are constrained to 0; $q_1 \equiv 0 \equiv q_r$. If this restriction is made in the prepotential $W$ of $D_r^{(1)}$ theory together with $2\beta \to \beta$ (and $r \to r+2$), it gives the prepotential of the $D_{r+1}^{(2)}$ to be discussed shortly in section 3.2.3. The corresponding eigenfunction is (3.38). The formula (3.32) also 'explains' the non-existence of the corresponding eigenfunction in $B_r^{(1)}$ theory, which is obtained by restriction $q_r \equiv 0$ (together with $r \to r+1$).

For the special case of $r = 3$ the eigenvalues for the spinor and anti-spinor weights are lower than that of the vector weights. We find several lower lying eigenstates:

$D_3^{(1)}$:

$$\phi_{s1} = \sin q_1 \sin q_2 \sin q_3 \qquad \tilde{H}\phi_{s1} = (4\beta+3/2)\phi_{s1} \tag{3.33}$$

$$\phi_{s2} = \cos q_1 \cos q_2 \cos q_3 \qquad \tilde{H}\phi_{s2} = (4\beta+3/2)\phi_{s2} \tag{3.34}$$

$$\phi_{ss} = \sin 2q_1 \sin 2q_2 \sin 2q_3 \qquad \tilde{H}\phi_{ss} = (8\beta+6)\phi_{ss} \tag{3.35}$$

$$\phi_2 = \cos 2q_1 \cos 2q_2 + \cos 2q_1 \cos 2q_3 + \cos 2q_2 \cos 2q_3 + 2\beta/(1+2\beta)$$
$$\tilde{H}\phi_2 = (8\beta+4)\phi_2. \tag{3.36}$$

These are closely related to the eigenfunctions of the $A_r^{(1)}$, (3.26), (3.27) and (3.29) since $A_3^{(1)} \cong D_3^{(1)}$.

*3.2.3.* $D_{r+1}^{(2)}$. The extended Dynkin diagram of $D_{r+1}^{(2)}$, figure 3, can be obtained from that of $B_{r+1}^{(1)}$ by folding the left 'fish tail' containing the affine simple root. Then $B_{r+1}^{(1)}$ is obtained from $D_{r+2}^{(1)}$ by folding the right 'fish tail' corresponding to the spinor and anti-spinor weights. The affine simple root of $D_{r+1}^{(2)}$ is the 'lowest short root' of $B_r$. In this case the spectrum of the small oscillations, up to the coupling constant $\beta$, is

$$4\{1, \ldots, \sin^2(j\pi/(r+1))/\sin^2(\pi/(r+1)), \ldots\}. \tag{3.37}$$


**Figure 4.** $C_r^{(1)}$ Dynkin diagram with the numbers $n_j$ attached. The black spot is the affine simple root.

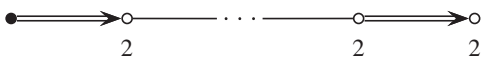


**Figure 5.** $A_{2r}^{(2)}$ Dynkin diagram with the numbers $n_j$ attached. The black spot is the affine simple root.

The lowest eigenvalue is an integer 4 (times $\beta$) which corresponds to the vector weights of $B_r$, the leftmost white vertex in figure 3. The set of vector weights is $\mathbf{V} = \{\pm\mathbf{e}_j | j = 1, \ldots, r\}$. Let us introduce the corresponding wavefunctions

$$\phi = \phi_{\mathbf{V}} + 2\beta/(1 + 2\beta) \qquad \phi_{\mathbf{V}} = \sum_{\mu\in\mathbf{V}} \mathrm{e}^{2\mathrm{i}\mu\cdot q} = 2\sum_{j=1}^{r} \cos 2q_j. \tag{3.38}$$

It is easy to see

$$\tilde{H}\phi = (4\beta + 2)\phi \qquad -\tfrac{1}{2}\triangle\phi_{\mathbf{V}} = 2\phi_{\mathbf{V}}. \tag{3.39}$$

*3.2.4.* $C_r^{(1)}$. The spectrum of the small oscillations, up to the coupling constant $\beta$, is easily read from table 1

$$8\{1, \ldots, \sin^2(j\pi/(2r))/\sin^2(\pi/(2r)), \ldots\}. \tag{3.40}$$

The lowest eigenvalue is an integer 8 (times $\beta$) which belongs to the vector weight of $C_r$, corresponding to the leftmost white vertex of figure 4. The set of vector weights is $\mathbf{V} = \{\pm\mathbf{e}_j | j = 1, \ldots, r\}$. Let us introduce the corresponding wavefunctions

$$\phi_{\mathbf{V}} = \sum_{\mu\in\mathbf{V}} \mathrm{e}^{2\mathrm{i}\mu\cdot q} = 2\sum_{j=1}^{r} \cos 2q_j. \tag{3.41}$$

It is easy to see

$$\tilde{H}\phi_{\mathbf{V}} = (8\beta + 2)\phi \qquad -\tfrac{1}{2}\triangle\phi_{\mathbf{V}} = 2\phi_{\mathbf{V}}. \tag{3.42}$$

As is well known $C_r^{(1)}$ is obtained from $A_{2r-1}^{(1)}$ by folding. The above eigenfunction originates from (3.25).

*3.2.5.* $A_{2r}^{(2)}$. This is also called a $BC_r$ root system, which is obtained by adding the affine root of $C_r^{(1)}$ to the set of simple roots of $B_r$ (see figure 5). The spectrum of the small oscillations, up to the coupling constant $\beta$ is

$$8\{1, \ldots, \sin^2(j\pi/(2r + 1))/\sin^2(\pi/(2r + 1)), \ldots\}. \tag{3.43}$$

The lowest eigenvalue is an integer 8 (times $\beta$) which corresponds to the vector weight of $B_r$, $\mathbf{V} = \{\pm\mathbf{e}_j | j = 1, \ldots, r\}$. Let us introduce the corresponding wavefunctions

$$\phi = \phi_{\mathbf{V}} + 4\beta/(1 + 4\beta) \qquad \phi_{\mathbf{V}} = \sum_{\mu\in\mathbf{V}} \mathrm{e}^{2\mathrm{i}\mu\cdot q} = 2\sum_{j=1}^{r} \cos 2q_j. \tag{3.44}$$

It is easy to see

$$\tilde{H}\phi = (8\beta + 2)\phi \qquad -\tfrac{1}{2}\triangle\phi_{\mathbf{V}} = 2\phi_{\mathbf{V}}. \tag{3.45}$$

As in the $D_{r+1}^{(2)}$ case (3.38), the eigenfunction (3.44) has a constant part. This is related to the fact that the vector representation of $B_r$ contains a zero weight. In contrast, the vector representation of $C_r$ does not contain a zero weight and the corresponding eigenfunction (3.41) does not have a constant part. This also explains why the eigenfunctions corresponding to the vector and conjugate vector representations (3.26) do not have a constant part, whereas that corresponding to the adjoint representation (3.27) has a constant part. The adjoint representation has a rank number of zero weight.

*3.2.6. Exceptional affine Lie algebras.* For $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ and $F_4^{(1)}$ none of the frequencies of (3.20) are integers. The $G_2^{(1)}$ case, which is obtained from $D_4^{(1)}$ by threefold folding, has two integer eigenvalues $\{8, 24\}$ inherited from $D_4^{(1)}$. As shown in section 3.2.2, we found no exact eigenfunctions for $D_r^{(1)}$ and $D_4^{(1)}$. Therefore we do not expect any exact eigenfunctions for the exceptional affine Toda–Sutherland systems and we have got none.

### *3.3. Comments on non-affine Toda–Sutherland systems*

In the Toda molecule (Toda field theory) interactions, the affine simple root $\alpha_0$ plays an essential role for the existence of an equilibrium. However, with $1/\sin^2 q$ type interactions, an equilibrium is achieved without the affine simple root $\alpha_0$. This opens a way to consider (non-affine) Toda–Sutherland systems characterized by a prepotential

$$W(q) = \beta \sum_{\alpha\in\Pi} \log|\sin(\alpha\cdot q)| = \beta \sum_{j=1}^{r} \log|\sin(\alpha_j\cdot q)|. \tag{3.46}$$

Note that it does not contain the affine simple root $\alpha_0$ nor the Dynkin–Kac labels $\{n_j\}$. Since the highest root is not contained in the prepotential, the configuration space now is

$$\mathrm{PW}_N = \{q\in\mathbf{R}^r | 0 < \alpha\cdot q < \pi, \quad \alpha\in\Pi\}. \tag{3.47}$$

Finding the equilibrium position $\bar{q}$ of the classical potential is easy. It is again proportional to the Weyl vector $\varrho$ (2.9)

$$\bar{q} = \frac{\pi}{2}\varrho \qquad \alpha_j\cdot\bar{q} = \frac{\pi}{2} \qquad \cot[\alpha_j\cdot\bar{q}] = 0 \quad \Rightarrow \quad \left.\frac{\partial W(q)}{\partial q_j}\right|_{\bar{q}} = 0 \qquad j = 1,\dots,r. \tag{3.48}$$

Due to the linear independence of the simple roots, this equilibrium is unique in the configuration space (3.47). The frequencies of small oscillations near the equilibrium are the eigenvalues of the matrix

$$\widetilde{W}_{jk} = -\left.\frac{\partial^2 W(q)}{\partial q_j \partial q_k}\right|_{\bar{q}} = \beta\sum_{l=1}^{r}(\alpha_l)_j(\alpha_l)_k = \beta\tilde{M}_{jk} \tag{3.49}$$

$$\tilde{M} = \sum_{j=1}^{r}\alpha_j\otimes\alpha_j. \tag{3.50}$$

For the simply laced root systems $(A, D, E)$ the spectrum of $\tilde{M}$ is the same as the spectrum of the Cartan matrix $C_{jk} = 2\alpha_j\cdot\alpha_k/\alpha_k^2$, $j, k\in\Pi$. There is a universal formula for the spectrum of $\tilde{M}$ for the $A, D, E$ series:

$$\mathrm{Spec}(\tilde{M}) = \{4\sin^2(e_1/2h),\dots,4\sin^2(e_r/2h)\} \qquad e_1,\dots,e_r\text{: exponents.} \tag{3.51}$$

The exponents of simply laced root systems are given in table 2.

**Table 2.** The exponents $e_j$ for simply laced root systems.

| $\Delta$ | $h$ | Exponents, $e_1, \ldots, e_r$ | $\Delta$ | $h$ | Exponents, $e_1, \ldots, e_r$ |
|---|---|---|---|---|---|
| $A_r$ | $r+1$ | $1, 2, 3, \ldots, r$ | $E_6$ | 12 | 1, 4, 5, 7, 8, 11 |
| $D_r$ | $2r$ | $1, 3, 5, \ldots, 2r-1; r-1$ | $E_7$ | 18 | 1, 5, 7, 9, 11, 13, 17 |
| | | | $E_8$ | 30 | 1, 7, 11, 13, 17, 19, 23, 29 |

For the $B$ series and $G_2$ we have

$$B_r: \quad \mathrm{Spec}(\tilde{M}) = \{4\sin^2(2j - 1/2(2r+1)) | j = 1, \ldots, r\} \tag{3.52}$$

$$G_2: \quad \mathrm{Spec}(\tilde{M}) = \{(4 - \sqrt{13})/3, (4 + \sqrt{13})/3\} \tag{3.53}$$

and analytical formulae are not known for the entire spectrum of $\tilde{M}$ in $C_r$ and $F_4$.

Although some of the eigenfrequencies of the small oscillations near the classical equilibrium (3.51), (3.52) are integers, they are definitely not the lowest lying ones. According to the quantum–classical correspondence [21], we do not expect to find the exact eigenfunctions for the lowest lying states, which have non-integer eigenvalues. Thus it is highly unlikely that the eigenfunctions for the higher excited states, being orthogonal to all the lower lying ones, could be obtained exactly, even for those belonging to integer classical eigenvalues. In fact we have not been able to find any exact eigenfunctions for the (non-affine) Toda–Sutherland systems (3.46).

## 4. Affine Toda–Calogero systems

Like the affine Toda–Sutherland system, the affine Toda–Calogero system can be defined for any affine root system $\Pi_0$ (2.6). However, in many respects the affine Toda–Calogero systems have less remarkable properties than the affine Toda–Sutherland systems discussed in the preceding section. The equilibrium position $\bar{q}$ does not have a simple characterization. Except for the systems based on the $A^{(1)}$ series, the small oscillations near the equilibrium do not have *integer* (times the coupling constant) eigenvalues other than 2, which is universal for all the potentials with quadratic plus inverse quadratic dependence on the coordinate $q$ [23].

The prepotential of the affine Toda–Calogero system is obtained from that of the affine Toda–Sutherland system (3.1) by changing $\sin \alpha_j \cdot q \to \alpha_j \cdot q$ and adding a harmonic confining potential with (angular) frequency $\omega > 0$:

$$W(q) = \beta \sum_{j=0}^{r} n_j \log |\alpha_j \cdot q| - \frac{\omega}{2} q^2. \tag{4.1}$$

Because of the singularity of the potential we restrict the configuration space to the *principal Weyl chamber* for simplicity:

$$\mathrm{PW} = \{q \in \mathbf{R}^r | \alpha \cdot q > 0, \quad \alpha \in \Pi\}. \tag{4.2}$$

(Due to the non-invariance under the Weyl group, theories with different configuration spaces are physically different. For example, they have different (non-equivalent) equilibrium positions.) The classical and quantum Hamiltonians are given in terms of the prepotential $W$ by the same formulae (3.2) and (3.3). The classical equilibrium position $\bar{q}$ is determined by

$$\frac{\partial W(\bar{q})}{\partial q_k} = 0 \qquad k = 1, \ldots, r \quad \Longleftrightarrow \quad \beta \sum_{j=0}^{r} \frac{n_j \alpha_j}{\alpha_j \cdot \bar{q}} = \omega \bar{q}. \tag{4.3}$$

In contrast to the Calogero systems in which $\bar{q}$ corresponds to the zeros of classical polynomials, i.e. the Hermite and the Laguerre polynomials for the classical root systems [22, 16], the present case does not have such simple characterization. The frequencies of small oscillations near the equilibrium are given by the eigenvalues of the matrix

$$\widetilde{W} = \text{Matrix}\left(-\frac{\partial^2 W(\bar{q})}{\partial q_j \partial q_k}\right).$$

We have evaluated $\bar{q}$ and $\widetilde{W}$ numerically for various affine root systems. We will discuss the systems based on the $A^{(1)}$ series in section 4.1. In all the other cases the only integer (times $\omega$) eigenvalue of $\widetilde{W}$ is 2, which exists in all the cases based on any root system. In fact it is more universal and exists for all the potentials with quadratic ($q^2$) plus inverse quadratic dependence on the coordinate $q$ [23, 9] without any root or weight structure. This eigenvalue 2 gives rise to exact quantum eigenfunctions $\phi_n(q)$ which are proportional to the Laguerre polynomial [23, 9] in $q^2$,

$$\tilde{H}\phi_n(q) = 2\omega n\phi_n(q) \qquad \phi_n(q) \propto L_n^{(E_0/\omega-1)}(\omega q^2) \qquad n = 1, 2, \ldots \tag{4.4}$$

in which $E_0 = (\beta h + r/2)\omega$ is the ground state energy and $h$ is the Coxeter number (2.12). Let us emphasize that these quantum eigenfunctions are also universal in the above sense.

Here the similarity transformed Hamiltonian $\tilde{H}$ and the eigenfunctions $\{\phi_n(q)\}$ are defined in terms of the ground state wavefunction $\psi_0 = \mathrm{e}^W$ in the same formulae as before (3.21)–(3.23).

### *4.1.* $A_r^{(1)}$

This theory and its possible generalization have been discussed rather extensively by Khare and collaborators [13–15] with explicit forms of quantum eigenfunctions. These multi-particle dynamics have nearest and next-to-nearest interactions with rational $1/q^2$ plus $q^2$ potentials. Here we discuss the relationship between the exact eigenfunctions and their classical counterparts [21]. The $A_2^{(1)}$ affine Toda–Calogero system is identical with the $A_2$ Calogero system. The spectrum of $\widetilde{W}$ for $A_r^{(1)}$, $r \geqslant 3$ has a form

$$\text{Spec}(\widetilde{W}) = \omega\{1, 2, 3, *, \ldots\} \tag{4.5}$$

in which $*, \ldots$ denote non-integers greater than 3.

The interpretation of these three integer eigenvalues is quite clear. The lowest one corresponds to the elementary excitation of the *centre of mass* coordinates $Q = q_1 + \cdots + q_{r+1}$ and the quantum eigenfunction belonging to the eigenvalue $n\omega$ is essentially the Hermite polynomial of degree $n$ in $Q$. The eigenfunctions corresponding to the eigenvalue 2 are the Laguerre polynomials (4.4) mentioned above. Let us introduce the elementary symmetric polynomial of degree $k$ in $q_1, \ldots, q_{r+1}$ [21]:

$$\prod_{j=1}^{r+1}(x + q_j) = \sum_{k=0}^{r+1} S_k x^{r+1-k} \qquad S_0 = 1 \quad S_1 = q_1 + \cdots + q_{r+1} \equiv Q. \tag{4.6}$$

Since $S_k$ is annihilated by the Laplacian, $\triangle S_k = 0$, one finds easily the exact quantum eigenfunction $\phi_3$ corresponding to the integer eigenvalue 3 in (4.5):

$$\tilde{H}S_3 = 3\omega S_3 + \beta(r-1)Q \qquad \tilde{H}\phi_3 = 3\omega\phi_3 \qquad \phi_3 = S_3 + \frac{\beta(r-1)}{2\omega}Q. \tag{4.7}$$

## 5. Summary and comments

The affine Toda–Sutherland system is introduced for any affine root system as a cross between the affine Toda molecule and the Sutherland system. That is, the potential is trigonometric, $1/\sin^2 q$, and the multi-particle interactions are governed by the affine simple roots only, in contrast to the entire set of roots in the Sutherland system. It has remarkable universal features. The classical equilibrium point is $\pi\varrho/h$ ($\varrho$: Weyl vector, $h$: Coxeter number) and the frequencies of small oscillations near the equilibrium are proportional to the corresponding affine Toda masses. In most cases based on classical affine Lie algebras, some low lying frequencies are integers (times a coupling constant). They give rise to exact quantum eigenvalues and eigenfunctions. The ground state eigenfunctions are always given explicitly. Thus the affine Toda–Sutherland systems provide examples of a new type of *quasi-exactly solvable* multi-particle dynamics.

Affine Toda–Calogero systems with rational ($1/q^2$ plus $q^2$) potentials are found to be less remarkable than their trigonometric counterparts. They possess an infinite number of exact eigenvalues and eigenfunctions which are well known. We have shown that the affine Toda–Calogero systems based on $A^{(1)}$ series have three lowest frequencies $\omega$, $2\omega$ and $3\omega$ of small oscillations near the classical equilibrium. They all correspond to exact quantum eigenvalues and eigenfunctions.

It would be interesting to understand these 'partially integrable' affine Toda–Sutherland–Calogero systems from various points of view: relationship with the random matrix models, analysis from the regular and chaotic dynamics, etc.

In [13–15] many interesting multi-particle dynamics, rational and trigonometric, related to the root systems of $B_r$, $C_r$, $BC_r$ and $D_r$, were introduced. They resemble our affine Toda–Sutherland and affine Toda–Calogero systems but they cannot be characterized in terms of affine simple roots. Unified understanding of these systems is wanted.

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